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ON THE STRESSES IN AN ELASTIC–PLASTIC ANNULAR DISK OF VARIABLE THICKNESS UNDER EXTERNAL PRESSURE

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Abstract—The plane state of stress in an elastic–plastic annular disk of variable thickness subjected to external pressure is studied. The thickness of the annular disk is assumed to vary exponentially with any power of the radius. The analysis is based on Tresca's yield condition, its associated flow rule and linear strain hardening. The stresses are obtained in terms of confluent hypergeometric functions.

NOTATION

σ_r, σ_θ	radial and circumferential stress components
σ_0, σ_y	initial and subsequent yield stress respectively
ε_{eq}	equivalent plastic strain
$d\varepsilon$	strain increment
u	radial displacement
E, ν	Young's modulus and Poisson ratio
η	work hardening parameter
r_i, r_d	inner and outer radii of the disk
z	elastic–plastic interface radius
p_0	external pressure
q	r_i/r_d radius ratio
h	local thickness
h_0, k	real constants in $h = h_0\,e^{1-(r/r_d)^k}$
A, B, C, D	constants of integration
p	superscript denoting plastic component.

It is convenient to introduce the following dimensionless quantities:

$$x = \frac{r}{r_d}, \quad \rho = \frac{z}{r_d}, \quad H = \frac{\eta\sigma_0}{E}, \quad \bar{\sigma}_{ij} = \frac{\sigma_{ij}}{\sigma_0}, \quad \bar{u} = \frac{Eu}{\sigma_0 r_d}.$$

1. INTRODUCTION

The first modern treatment for an elastic–plastic annular disk with linear hardening subjected to external pressure has been given by Gamer (1983, 1984). The above analyses consider the thickness of the disk as constant. However, it is well known that disks with variable thickness are frequently found in mechanical engineering. Axial symmetric solutions of the isotropic and orthotropic disks, including variable thickness, variable density, and other cases, can be found in most of the standard elasticity textbooks. The elastic–plastic annular disk of variable thickness subjected to external pressure has been solved by this author (1992) by assuming the thickness function in the form $h = h_0(r/r_d)^{-n}$. In the present work we consider the same problem for a more general form of the thickness function $h = h_0\,e^{1-(r/r_d)^k}$ where h_0 and k are real constants (Fig. 1). It is the purpose of this

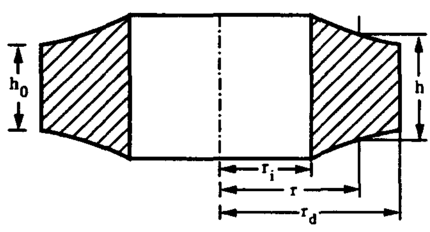

Fig. 1. Disk geometry, $h(r) = h_0\,e^{1-(r/r_d)^k}$.

work to present a closed form solution of an elastic–plastic annular disk subjected to external pressure for which a radial thickness variation is assumed.

2. BASIC EQUATIONS AND SOLUTION

We consider a state of plane stress and assume infinitesimal deformation. It is assumed that the variation of thickness is radial and symmetric with respect to the midplane. The analysis is based on Tresca's yield condition, its associated flow rule and linear strain hardening.

In the plastic region, we shall confine ourselves by assuming σ_θ is the largest in magnitude and σ_r intermediate, i.e. $\sigma_z(=0) > \sigma_r > \sigma_\theta$. Hence, the Tresca yield condition adopts the form

$$\sigma_\theta = -\sigma_y. \tag{1}$$

If the work hardening law is taken to be

$$\sigma_y = \sigma_0(1+\eta\varepsilon_{eq}), \tag{2}$$

where σ_0 is the initial tensile yield stress, η is the hardening parameter and ε_{eq} is an equivalent plastic strain. According to the flow rule associated to the Tresca yield condition

$$d\varepsilon_r^p = 0, \quad d\varepsilon_\theta^p + d\varepsilon_z^p = 0. \tag{3}$$

Consideration of the equivalence of the increment of plastic work yields

$$\varepsilon_{eq} = -\varepsilon_\theta^p. \tag{4}$$

The governing differential equation of equilibrium [see, for example, Timoshenko and Gooodier (1970)] is

$$\frac{d}{dr}(hr\sigma_r) - h\sigma_\theta = 0, \tag{5}$$

where h is the thickness function, and geometric relations

$$\varepsilon_r = \frac{du}{dr}, \quad \varepsilon_\theta = \frac{u}{r} \tag{6}$$

hold in the annular disk irrespective of material behavior. Total strains are decomposed into elastic and plastic components.

The stress–strain relations are:

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) + \varepsilon_r^p, \quad \varepsilon_r^p = 0, \tag{7}$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) + \varepsilon_\theta^p. \tag{8}$$

Since we restrict ourselves to small strains, ε_r and ε_θ must satisfy the compatibility equation

$$\frac{d}{dr}(r\varepsilon_\theta) = \varepsilon_r. \tag{9}$$

Substituting the strains ε_r and ε_θ in the compatibility equation (9), using (1)–(5) we obtain

$$r^2\frac{d^2\sigma_r}{dr^2}+r\left(3+r\frac{h'}{h}\right)\frac{d\sigma_r}{dr}+\left[\frac{1}{H+1}+\left(v\frac{H}{H+1}+2\right)r\frac{h'}{h}+\left(\frac{h''}{h}-\frac{h'^2}{h^2}\right)r^2\right]\sigma_r=\frac{-\sigma_0}{H+1}, \tag{10}$$

which is the differential equation expressed in terms of the radial stress, where a prime denotes differentiation with respect to r and $H = \eta\sigma_0/E$.

The thickness of the annular disk is assumed to vary along the radius in the form

$$h = h_0\,\mathrm{e}^{1-(r/r_d)^k}, \tag{11}$$

where h_0 and k are real constants.

Using (11), the differential equation (10) expressed in terms of the radial stress can be rewritten in the form

$$r^2\frac{d^2\sigma_r}{dr^2}+\left[3-k\left(\frac{r}{r_d}\right)^k\right]r\frac{d\sigma_r}{dr}+\left[\frac{1}{H+1}-\left(1+k+\frac{vH}{H+1}\right)k\left(\frac{r}{r_d}\right)^k\right]\sigma_r=-\frac{\sigma_0}{H+1}. \tag{12}$$

By using the transformation

$$r_1 = (r/r_d)^k \tag{13}$$

we get the following differential equation of confluent hypergeometric type (It and Slater, 1960):

$$\frac{d^2\sigma_r}{dr_1^2}+\left[\left(1+\frac{2}{k}\right)\frac{1}{r_1}-1\right]\frac{d\sigma_r}{dr}+\left[\frac{1}{k^2(H+1)}\frac{1}{r_1^2}-\frac{(1+k+vw^2)}{k}\frac{1}{r_1}\right]\sigma_r = -\frac{\sigma_0}{k^2(H+1)}\frac{1}{r_1^2}, \tag{14}$$

where $w^2 = H/(H+1)$.

The general solution of this equation may be written as follows:

$$\bar{\sigma}_r = \bar{A}x^{w-1}F(a;b;x^k)+\bar{B}x^{-(w+1)}F(1+a-b;2-b;x^k)+\bar{R}_{\text{part}} \tag{15}$$

and using the equilibrium equation, the circumferential stress is found to be

$$\begin{aligned}\bar{\sigma}_\theta = {} & \bar{A}\left[wx^{w-1}F(a;b;x^k)+\frac{a}{b}kx^{k+w-1}F(1+a;1+b;x^k)\right]\\ & +\bar{B}\left[\frac{1+a-b}{2-b}kx^{k-w-1}F(2+a-b;3-b;x^k)-wx^{-(w+1)}F(1+a-b;2-b,x^k)\right]\\ & +\frac{d}{dx}(x\bar{R}_{\text{part}})-kx^k[\bar{A}x^{w-1}F(a;b;x^k)+\bar{B}x^{-w-1}F(1+a-b;2-b;x^k)+\bar{R}_{\text{part}}],\end{aligned} \tag{16}$$

where

$$a = 1+w(1+vw)/k,\quad b = 1+(2w/k),\quad x = r/r_d,$$
$$\bar{\sigma}_r = \sigma_r/\sigma_0,\quad \bar{\sigma}_\theta = \sigma_\theta/\sigma_0,\quad \bar{R}_{\text{part}} = R_{\text{part}}/\sigma_0,$$

F is a confluent hypergeometric function and R_{part} is a particular integral of the solution. The particular solution can be found in the series form (Sneddon, 1961) as follows:

$$\bar{R}_{\text{part}} = \bar{A}_0/r_1 + \bar{A}_1/r_1^2 + \bar{A}_2/r_1^3 + \bar{A}_3/r_1^4 + \cdots \bar{A}_n/r_1^{n+1} + \cdots,$$

where the first and general terms are:

$$\bar{A}_0 = \frac{1}{k(H+1)(1+\nu w^2)}, \quad \bar{A}_n = \frac{\dfrac{1}{k^2(H+1)} - n\left(\dfrac{2}{k} - n\right)}{\dfrac{1}{k}(1+\nu w^2) - n} \bar{A}_{n-1}.$$

Hence, the particular solution may be written as follows:

$$\bar{R}_{\text{part}} = \frac{1}{k(H+1)(1+\nu w^2)} \left\{ \frac{1}{r_1} + \frac{\dfrac{1}{k^2(H+1)} - \left(\dfrac{2}{k} - 1\right)}{[(1+\nu w^2)/k] - 1} \frac{1}{r_1^2} + \frac{\left\langle \dfrac{1}{k^2(H+1)} - 2\left(\dfrac{2}{k} - 2\right)\right\rangle \left\langle \dfrac{1}{k^2(H+1)} - \left(\dfrac{2}{k} - 1\right)\right\rangle}{\langle [(1+\nu w^2)/k] - 2 \rangle \langle [(1+\nu w^2)/k] - 1 \rangle} \frac{1}{r_1^3} + \cdots \right\}. \quad (17)$$

Substituting for σ_r and σ_θ in (8), and using (1)–(4), and the circumferential strain–radial displacement relation, the radial displacement is obtained as follows:

$$\bar{u} = \left\{ \frac{\bar{A}}{w^2} \left[(w - \nu w^2 - kx^k) x^{w-1} F(a; b; x^k) + \frac{a}{b} k x^{k+w-1} F(1+a; 1+b; x^k) \right] + \frac{\bar{B}}{w^2} \left[\frac{1+a-b}{2-b} k x^{k-w-1} F(2+a-b; 3-b; x^k) - (w + \nu w^2 + kx^k) x^{-w-1} F(1+a-b; 2-b; x^k) \right] + \frac{1}{w^2} \left[\frac{\mathrm{d}}{\mathrm{d}x} (x \bar{R}_{\text{part}}) - kx^k \bar{R}_{\text{part}} \right] - \nu \bar{R}_{\text{part}} + \frac{1}{H} \right\} x, \quad (18)$$

where $\bar{u} = Eu/\sigma_0 r_d$.

In the elastic region, $z < r \leqslant r_d$ the stresses and radial displacement are obtained from eqns (15), (16) and (18) for $H \to \infty$ as follows:

$$\bar{\sigma}_r = \bar{C} F(a_1; b_1; x^k) + \bar{D} x^{-2} F(1+a_1-b_1; 2-b_1; x^k), \quad (19)$$

$$\bar{\sigma}_\theta = \bar{C} \left[F(a_1; b_1; x^k) + k \frac{a_1}{b_1} x^k F(1+a_1; 1+b_1; x^k) \right] + \bar{D} \left[-x^{-2} F(1+a_1-b_1; 2-b_1; x^k) + k \frac{1+a_1-b_1}{2-b_1} x^{k-2} F(2+a_1-b_1; 3-b_1; x^k) \right] - kx^k \left[\bar{C} F(a_1; b_1; x^k) + \bar{D} x^{-2} F(1+a_1-b_1; 2-b_1; x^k) \right], \quad (20)$$

$$\bar{u} = \left\{\bar{C}\left[(1-\nu)F(a_1;b_1;x^k)+k\frac{a_1}{b_1}x^kF(1+a_1;1+b_1;x^k)\right]\right.$$

$$+\bar{D}x^{-2}\left[-(1+\nu)F(1+a_1-b_1;2-b_1;x^k)+k\frac{1+a_1-b_1}{2-b_1}x^kF(2+a_1-b_1;3-b_1;x^k)\right]$$

$$\left.-kx^k[\bar{C}F(a_1;b_1;x^k)+\bar{D}x^{-2}F(1+a_1-b_1;2-b_1;x^k)]\right\}x, \quad (21)$$

where

$$a_1 = 1+\frac{1+\nu}{k}, \quad b_1 = 1+\frac{2}{k}.$$

3. THE ELASTIC–PLASTIC STRESS DISTRIBUTION

The above general expressions for stresses and displacements contain the unknown constants A, B, C and D. An additional unknown is the elastic–plastic interface radius z. For the determination of these five unknowns there are five conditions available. The most convenient ones are: $\sigma_r^p = 0$ at $r = r_i$, $\sigma_\theta^p = \sigma_\theta^e = -\sigma_0$, $\sigma_r^p = \sigma_r^e$, at $r = z$ and $\sigma_r^e = -P_0$ at $r = r_d$. When these conditions are enforced, the unknowns are found to be

$$\bar{B}\left\{[k_2\rho^{k-w-1}F(2+a-b;3-b;\rho^k)-(w+k\rho^k)\rho^{-(w+1)}F(1+a-b;2-b;\rho^k)]\right.$$

$$\left.-\frac{G_2}{G_1}[(w-k\rho^k)\rho^{w-1}F(a;b;\rho^k)+k_1\rho^{k+w-1}F(a+1;b+1;\rho^k)]\right\}$$

$$=\frac{R_0}{G_1}[(w-k\rho^k)\rho^{w-1}F(a;b;n\rho^k)+k_1\rho^{k+w-1}F(a+1;b+1;\rho^k)]$$

$$-\left[1+\left|\frac{\mathrm{d}}{\mathrm{d}x}(x\bar{R}_{\text{part}})-kx^k\bar{R}_{\text{part}}\right|_\rho\right], \quad (22)$$

$$\bar{A} = -\frac{1}{G_1}(R_0+G_2\bar{B}), \quad (23)$$

$$\bar{D}\left\{k_4\rho^{k-2}F(2+a_1-b_1;3-b_1;\rho^k)-(1+k\rho^k)\rho^{-2}F(1+a_1-b_1;2-b_1;\rho^k)\right.$$

$$\left.-\frac{G_4}{G_3}[k_3\rho^kF(1+a_1;1+b_1;\rho^k)+(1-k\rho^k)F(a_1;b_1;\rho^k)]\right\}$$

$$=\frac{\bar{p}_0}{G_3}[(1-k\rho^k)F(a_1;b_1;\rho^k)+k_3\rho^kF(1+a_1;1+b_1;\rho^k)]-1, \quad (24)$$

$$\bar{C} = -\frac{1}{G_3}(\bar{p}_0+G_4\bar{D}), \quad (25)$$

and the nondimensional elastic–plastic interface radius ρ can be found from the following equation:

$$\bar{B}\left[\rho^{-w-1}F(1+a-b;2-b;\rho^k)-\frac{G_2}{G_1}\rho^{w-1}F(a;b;\rho^k)\right]-\frac{R_0}{G_1}\rho^{w-1}F(a;b;\rho^k)+\bar{R}(\rho)_{\text{part}}$$

$$=\bar{D}\left[\rho^{-2}F(1+a_1-b_1;2-b_1;\rho^k)-\frac{G_4}{G_3}F(a_1;b_1;\rho^k)\right]-\frac{\bar{p}_0}{G_3}F(a_1;b_1;\rho^k), \quad (26)$$

where

$$k_1=\frac{ka}{b},\quad k_2=k\frac{1+a-b}{2-b},\quad k_3=\frac{ka_1}{b_1},\quad k_4=k\frac{1+a_1-b_1}{2-b_1},$$

$$G_1=q^{w-1}F(a;b;q^k),\quad G_2=q^{-w-1}F(1+a-b;2-b;q^k),$$

$$G_3=F(a_1;\mathrm{b}_1;1),\quad G_4=F(1+a_1-b_1;2-b_1;1),$$

$$R_0=\bar{R}(q)_{\text{part}},\quad \rho=\frac{z}{r_d},\quad q=\frac{r_i}{r_d},\quad \bar{p}_0=\frac{p_0}{\sigma_0}.$$

4. THE FULLY PLASTIC CASE

In the particular case, for $\rho = 1$ the annular disk becomes fully plastic. A and B constants can be determined from boundary conditions $\sigma_r^p = 0$ and $r = r_i$ and $\sigma_r^p = -P_0$ at $r = r_d$. When these conditions are enforced the unknowns are found to be:

$$\bar{B}\left[F(1+a-b;2-b;1)-q^{-2w}\frac{F(a;b;1)F(1+a-b;2-b;q^k)}{F(a;b;q^k)}\right]$$

$$=-\bar{P}_0-[|\bar{R}_{\text{part}}|_{x=1}-R_0q^{1-w}F(a;b;1)/F(a;b;q^k)], \quad (27)$$

$$\bar{A}=-\{R_0q^{1-w}+\bar{B}q^{-2w}F(1+a-b;2-b;q^k)\}/F(a;b;q^k). \quad (28)$$

5. NUMERICAL RESULTS AND DISCUSSION

The analysis of the elastic–plastic annular disks subjected to external pressure has been previously investigated by Gamer. Gamer's solution is very suitable for practical computations. Using the same basic approach the elastic–plastic annular disk of variable thickness subjected to external pressure has been solved by this author. However, the present work considers the variation of the thickness for a more intricate form $h = h_0\,\mathrm{e}^{1-(r/r_d)^k}$.

The present solution is not valid for the perfectly plastic material, i.e. $H = 0$. (In this case, $a = b = 1$.) However, it is possible to find an approximate solution from the present work for a very small value of the hardening parameter H. Alternatively, the solution for $H = 0$ can be found by substituting $\bar{\sigma}_\theta = -1$ everywhere in the equilibrium equation and solving for $\bar{\sigma}_r$.

Numerical results are presented graphically showing the influence of thickness parameter k and hardening parameter H on the distribution stresses, for $q = r_i/r_d = 0.5$ and $\nu = 1/3$.

Figure 2 shows the stress distributions in the fully plasticized annular disks of uniform and variable thickness, $k = 0$ and $k = 1.5$, respectively; in both cases, $H = 1/3$. By comparing the curves for $k = 0$ with $k = 1.5$, it is seen that, as a consequence of the nonuniform thickness, radial stress increases significantly at the outside boundary. In the fully plastic case, for $H = 0$, by using the equilibrium equation, limit external pressures for $k = 0$ and $k = 1.5$ are found to be $\bar{p}_0 = 0.5$ and $\bar{p}_0 = 0.70448$, respectively.

Figures 3 and 4 show the stress distributions in the elastic–plastic annular disks of uniform and variable thickness. In Figs 3 and 4, $k = 0.75$, $H = 1/3$ and $k = 0.8$, $H = 9/16$, respectively. In both cases, $\rho = 0.65$. By comparing the curves for Figs 3 and 4, it is seen

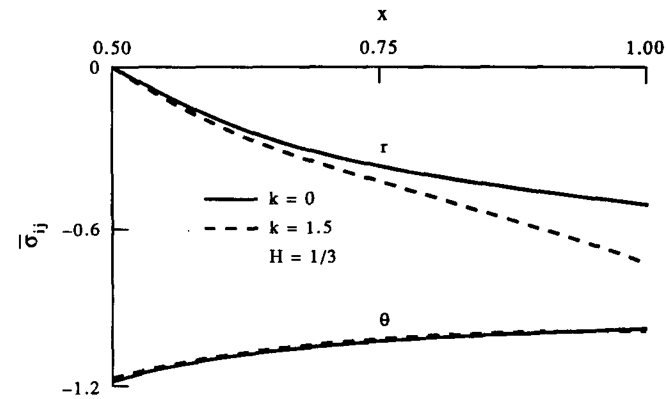

Fig. 2. Stress distribution in fully plasticized annular disks for $q = 0.5$, $k = 0$, $k = 1.5$ and $H = 1/3$.

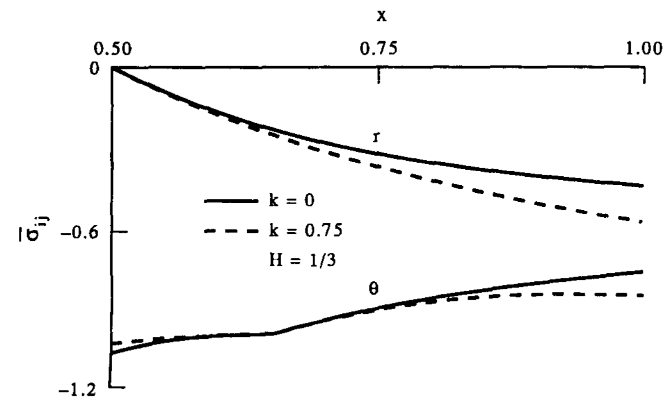

Fig. 3. Stress distribution in elastic–plastic annular disks for $q = 0.5$, $k = 0$, $k = 0.75$ and $H = 1/3$.

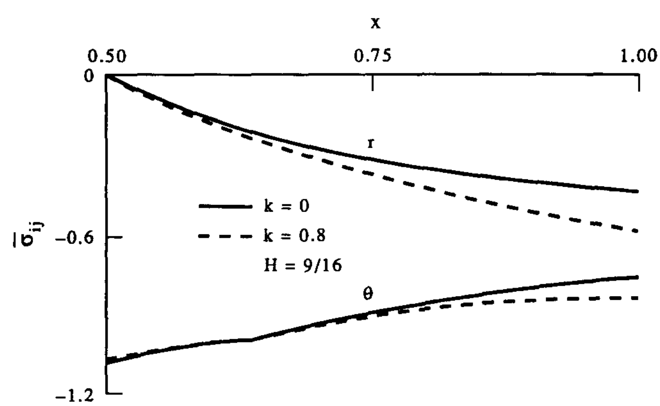

Fig. 4. Stress distribution in elastic–plastic annular disks for $q = 0.5$, $k = 0$, $k = 0.8$ and $H = 9/16$.

that, as a consequence of the nonuniform thickness, radial stress and circumferential stress at the outside boundary increase.

It is interesting to observe from the present analysis that the stresses in the fully plastic annular disk with variable thickness depend on the Poisson ratio ν. However it can be shown that the stresses in the fully plastic annular disk (Gamer, 1983) are not influenced by the Poisson ratio ν.

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